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Introduction to String Theory: Its Structure and its Uses [and Discussion]

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Introduction to string theory: its structure and its uses

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Over five years ago experiments at CERN confirmed that the weak (radioactive) interactions of elementary particles are mediated by gauge particles that are heavy relatives of the photon, namely the quantum of light and radio wave propagation. Gauge particles have to belong to a pattern given by the structure of a compact Lie group. Mathematicians listed such patterns at the beginning of the century and it seems that nature favours one of the 'exceptional' possibilities when nuclear forces are included.

Twenty years ago a picture of elementary particles as quantums of the excitations of a one-dimensional string was developed. Consistency with the principles of relativity and quantum mechanics seemed to require the aforementioned exceptional gauge structure as well as gravitational forces in Einstein's formulation. Thus a simple 'string' principle promised to explain and unify all the diverse fundamental forces of nature: electromagnetic, weak, nuclear and gravitational. Unfortunately, there remain detailed questions still to be resolved.

Nevertheless, the theory possesses rich mathematical structure encompassing Lie algebras and infinite-dimensional generalizations and complex algebraic geometry in a way which sheds valuable new perspectives on modern pure mathematics. At the same time it has unexpected applications in describing and classifying the modes of phase transition in two-dimensional materials, a classical problem in statistical physics.

An ideal introduction to string theory could, with some justice, be subtitled 'The string theory prerequisites for mathematics', but that would be too ambitious a title for what I have to say.

The reason I mention this is to draw attention to the fact that string theory is a theory of elementary particles and their fundamental interactions largely created in the decade after 1968 by physicists who were largely ignorant of modern advanced mathematics (some of which had not even been developed then). They certainly believed that they had a theory of considerable mathematical significance, but were unable to interest mathematicians in it at that time. Equally, they were unable to persuade the larger physics community of the relevance of their theories. Consequently, the theory went into hibernation only to reawaken dramatically when it was recognized that there was a rapprochement between string theory and newer ideas in particle physics, grand unification and the theory of anomalies, as well as ideas further afield, such as the representation theory of affine Kac–Moody algebras and the theory of Riemann surfaces and their moduli. Since then, progress has continued apace on several fronts, providing the occasion for this meeting.

String theory was, of course, always based on physical principles, but these have shifted as the structure has been clarified. As the original version was designed to include observed features of the scattering of nuclear particles, the physical principles were couched in terms of the scattering matrix and the presentation of the theory was algebraic in nature. Later it was

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realized that the theory applied rather to the fundamental forces between leptons and quarks rather than the forces between neutrons and protons and as a consequence the basic principles and the presentation of the theory became more geometric in nature. The result of this was that the same basic theory could be formulated in two apparently different ways, algebraic and geometric, and it is the comparison between these two approaches which has related different branches of mathematics in unexpected and fascinating ways, as described by other contributors to this Symposium.

Historically, the graph of the development of string theory has resembled a staircase with seemingly insurmountable obstacles suddenly overcome by a new insight, usually based on the appreciation of an unexpected manifestation of symmetry in a form subtly different from hitherto, for example, the role of the Virasoro algebra, super versions of it, and affine Kac–Moody algebras.

In motivating string theory I shall proceed directly to the current point of view, ignoring the history that can be gleaned from the previous Royal Society Discussion Meeting in 1969 or my talk at the 1974 London conference on particle physics (Olive 1974).

Hitherto the twentieth century has seen four new physical principles:

(i) Einstein's theory of special relativity (which unified space and time);

(ii) quantum mechanics;

(iii) the gauge principle, i.e. invariance under internal symmetries performed independently at different points of space and time;

(iv) gravity according to Einstein's theory of general relativity.

The simplest example of (iii) is a change in phase of the Schrödinger wave function for an electron;

$$\psi(x,t) \rightarrow e^{iq\chi(x,t)/\hbar}\psi(x,t). \tag{1}$$

It is a familiar feature of quantum mechanics that wave functions are complex, with an overall phase that is irrelevant. The phase change that occurs in (1) varies from point to point in space and time. Such phase changes evidently form a U(1) group at each point of space and time, called the gauge group. The demand that physics be independent of these 'gauge rotations' leads to the introduction of a 'gauge potential' or 'connection' satisfying Maxwell's equations, and hence the phenomena of electricity and magnetism, together with light- and radio-wave propagation. The constant q in the phase change is the electric charge of the electron. The most natural generalization of this group U(1) of unitary one by one matrices is to the group U(2) of unitary 2 by 2 matrices. The gauge theory of this yields a nonlinear generalization of Maxwell's equations satisfied by four gauge fields (corresponding to the four generators of the group U(2)). This successfully describes the phenomena of radioactivity in addition to electromagnetism, which corresponds to a U(1) subgroup of U(2) (not the invariant one). This is the Salam–Weinberg theory of electroweak interactions that was experimentally verified at CERN in 1982–83 by the detection of the three new gauge particles. The reason that detection was so difficult was that, owing to symmetry breaking effects, they were not massless like the photon, which is the particle of the U(1) gauge potential, but heavier than a hydrogen atom.

The success of this theory suggested that the nuclear forces be included similarly. It had already been realized that the forces between the quarks, the constituents of the protons and neutrons, were of the gauge type with group SU(3) (of 3 by 3 unitary matrices with unit determinant). The actual nuclear forces between the neutrons and protons were not of this

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type, being a complicated derived effect of the former. Because of the ubiquity of gauge forces an attractive goal would be a 'grand unification' whereby the gauge group was a simple Lie group containing SU(3) × U(2) as economically as possible. The most promising candidates turned out to be the following sequence of groups:

$$U(2) \times SU(3) \subset SU(5) \subset SO(10) \subset E_6 \subset E_7 \subset E_8. \tag{2}$$

The larger groups here are successively more approximate (and uncertain) owing to the symmetry-breaking effects. Nevertheless, once the matter particles, i.e. electrons, muons, neutrinos and quarks, are assigned to representations of these groups, usually what Bourbaki calls miniscule representations, we have an extremely succinct encapsulation of thousands of millions of pounds of experimental data. However, these statements are not the whole story: there are the symmetry-breaking effects to be studied in more detail by the future accelerators and the theoretical question of how to include gravity (iv). The results (2) add the stunning new question as to why nature has an apparent predilection for exceptional structures such as the E groups. The interest in string theory is that it promises successful answers to the latter two questions even though the question of symmetry breaking is still beyond it.

Not all theories of the gauge type are internally consistent when quantum mechanics is fully taken into account. The famous results from the late 1950s concerning the non-conservation of parity in weak (radioactive) interactions showed that left- and right-handed matter should be treated differently, as they transform under inequivalent representations of the grand unified gauge group (actually complex conjugates of each other in four dimensions of space and time, and hence complex). The trouble is that it is not always possible to quantize a given classical theory while preserving a symmetry because of what are called 'anomalies'. This can be seen from quantum mechanical action principle of Richard Feynman (1948):

$$\text{matrix element} = \iint \delta \,\text{fields}\, e^{i\,\text{action}/\hbar}. \tag{3}$$

In a conventional field theory the action is an integral over space-time of the lagrangian density that depends only on the fields and their derivatives evaluated at the relevant space-time point. In the classical limit of Planck's constant $\hbar$ being very small this gives the principle of stationary action, as it should, with local field equations. Symmetry in classical physics is guaranteed if the action is invariant, but the above shows that in quantum theory the integration measure over the fields must also be invariant. This is not automatic in the case of gauge symmetry when left-handed and right-handed matter transform according to inequivalent representations of the gauge group. (This effect is closely related to the Atiyah–Singer index theorem.) For the grand unified theories in four dimensions of space and time this problem is fortunately evaded by the choice of representations assigned to matter on the basis of the data.

The problem of anomalies will reappear in various guises when we have dealt with the next difficulty, which is the occurrence of divergences in the above expression, particularly when gravity is taken into account following Einstein's theory. Because we are dealing with local field equations we are necessarily imagining that the elementary particles occupy isolated points in space. This picture is rather singular especially when interaction is included. As time evolves, the particle point follows a 'world-line', depicting its trajectory in space and time with interactions occurring at the junctions of these world-lines; see figure 1.

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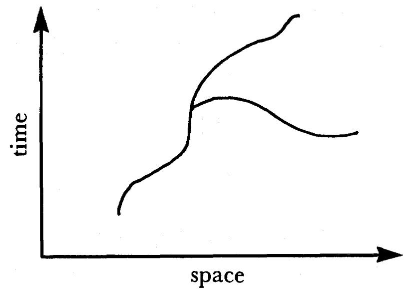

FIGURE 1

These junctions are highly singular points in this description and are the points where the divergences originate. Unfortunately, they appear to be an inevitable consequence of the principles (i)–(iii), which had, in other respects, been so successful. One lesson can be salvaged and that is that the action for freely moving point particles is beautifully simple and geometric:

$$\text{action} = -mc^2 \int d\tau, \tag{4}$$

where τ is the proper time of the particle, i.e. that measured by a clock travelling with it, and the integration is along the world-line of the particle. Thus the action is simply the length of the world-line and is stationary precisely for straight-line trajectories.

The string revolution started 20 years ago when it was realized that it was worth picturing elementary particles as occurring in families corresponding to the quantized modes of excitation of string. The original reason was to explain the spectrum of observed particles that included high spins, but later the motivation changed as we shall see. Thus if the string is open ended, I have as possible motions those depicted in figure 2.

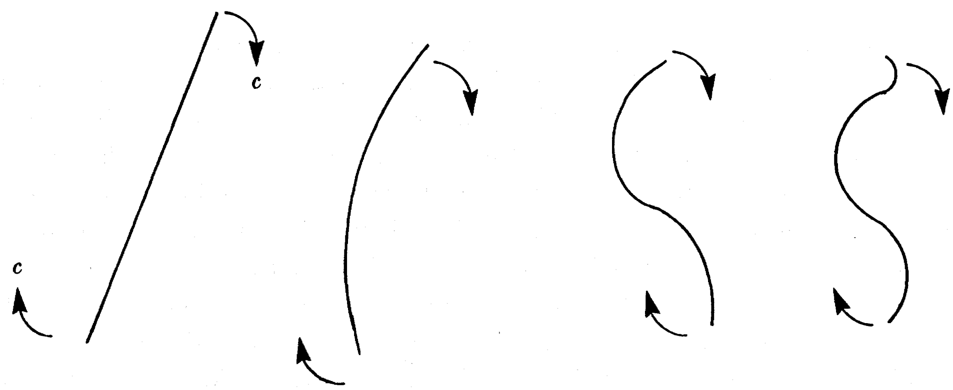

FIGURE 2

These illustrations indicate that the end-points of the string move at the speed of light. The frequency of the vibrations superimposed on the overall rotation are harmonics of a fundamental, just as they are for a vibrating violin string which has fixed end-points, even though the two boundary conditions are quite different.

These are classical solutions and so render stationary the action that again is extremely

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simple and geometric (Goto 1971; Y. Nambu, unpublished work). It is the two-dimensional 'area' of the world-sheet swathed out by the string as it moves in space and time. This is a natural generalization of the action (4), but, as we shall see, possesses more symmetry and intrinsic advantages compared with (4) in determining interactions:

$$\text{action} \propto \text{area of world-sheet} = \iint \mathrm{d}\tau\, \mathrm{d}\sigma \sqrt{(-\det h)}. \tag{5}$$

To say what we mean by area we need a metric on the world-sheet (supposing it to be a manifold) and we take the natural one imposed by its embedding in space and time that has the usual (flat) Lorentz metric $g^{\lambda\mu}$ appropriate to $D-1$ space variables and one time variable. Because of its geometric nature, the action (5) is independent of the choice of coordinates on the world-sheet. It is usual to suppose that the sheet possesses at least one tangent that is time-like at each point. $\tau = \tau^1$ is a time-like variable and $\sigma = \tau^2$ a space-like one. The coordinate in space-time of the point (τ, σ) is denoted $X^\lambda(\tau, \sigma)$. Then the metric is given by

$$h_{\alpha\beta} = \partial_\alpha X^\mu g_{\mu\nu} \partial_\beta X^\nu, \quad \text{where} \quad \partial_\alpha = \partial/\partial\tau^\alpha \quad \text{or} \tag{6a}$$

$$h = \begin{pmatrix} \dot{X}^2 & \dot{X}X' \\ \dot{X}X' & X'^2 \end{pmatrix} \tag{6b}$$

in matrix notation. Hence

$$-\det h = (\dot{X}X')^2 - \dot{X}^2 X'^2, \tag{7}$$

where dot and prime denote differentiations with respect to τ and σ, respectively.

There is an important feature of the action (5) that agrees with everyday experience of electromagnetic waves and so lends weight to the idea that (5) is the beginning of a theory encompassing the gauge theories previously discussed. This is called 'transversality'. The elementary particles are the quantized vibrations of the string harmonics previously mentioned. If we picture the vibrations in space-time, we see that the vibrations within the world-sheet have no meaning as they can be redefined away by a change of the variables σ and τ. This is good because we do not want vibrations in time. These would be difficult to interpret and lead to 'ghosts', which are incompatible with the principles of quantum mechanics. But it also follows that we can redefine away those space-like vibrations along the string, i.e. longitudinal vibrations. This leaves only $D-2$ meaningful directions transverse (or normal) to the world-sheet in which physical vibrations can occur.

When $D = 4$, our usual situation, this leaves two directions of transverse polarization or vibration. Light waves exhibit this feature by possessing two states of polarization orthogonal to the direction of propagation. This can be seen by superimposing two lenses of polarized sunglasses and rotating them relative to each other. Radio waves are also polarized transversally as is demonstrated by the fact that radio aerials on cars have to be mounted vertically.

This transversality plays an important role in the mathematical structure of string theory. It is related to the reparametrization invariance of the string action (5) and this recalls general covariance in Einstein's general relativity. I now explore the idea that the action (5) can be formulated in a way more like that theory.

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From that point of view, I think of the X^μ as D scalar fields on the world-sheet furnished with an SO$(D-1,1)$ internal symmetry. This suggests I consider an alternative action to (5):

$$\text{action} = \iint \mathrm{d}\sigma\,\mathrm{d}\tau\,\surd(-\det\tilde{h})\,\partial_\alpha X^\mu g_{\mu\nu}\partial_\beta X^\nu \tilde{h}^{\alpha\beta} \tag{8a}$$

$$= \iint \mathrm{d}\sigma\,\mathrm{d}\tau\,\surd(-\det\tilde{h})\,h_{\alpha\beta}\tilde{h}^{\alpha\beta} \tag{8b}$$

by using (6*a*) for the embedded metric. As in general relativity, I now treat the dummy metric $\tilde{h}_{\alpha\beta}$ as an independent variable in the action, in addition to the X^λ, which were the previous independent variables. There are consequently two Euler–Lagrange equations resulting from rendering (8) stationary to variations first of $\tilde{h}$:

$$\theta_{\alpha\beta} = 0, \tag{9}$$

where
$$\theta_{\alpha\beta} = \partial_\alpha X^\mu g_{\mu\nu}\partial_\beta X^\nu - \tfrac{1}{2}\tilde{h}_{\alpha\beta}h^{\gamma\delta}\tilde{h}_{\gamma\delta}, \tag{10}$$

and then of X^λ
$$\Delta X^\lambda = \partial_\alpha(\surd(-\det\tilde{h})\,\tilde{h}^{\alpha\beta}\partial_\beta X^\lambda) = 0. \tag{11}$$

Expression (10) is interpreted as the energy–momentum tensor within the string world-sheet. By (9), it vanishes, expressing the fact that there is no observable flow of energy within the world-sheet. This is another statement of transversality. It also follows from (9) that the dummy and induced metrics $\tilde{h}$ and h are proportional. The action (8*b*) is independent of the overall scale in $\tilde{h}$ and, as a consequence, (8*b*) reduces to the Nambu–Goto action (5). The action (8) is usually known as the Polyakov action though it is not due to him. The second Euler–Lagrange equation (11) states that X^λ satisfies the covariant Laplace equation on the world-sheet. If τ and σ are chosen to be orthonormal in the sense that (6*b*) reads as

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

this reduces to the wave equation, thus explaining the wave solutions described in figure 2.

The symmetry of the action (8) under rescaling of $\tilde{h}$, the dummy metric, is a new feature compared with the Nambu–Goto action (5). The process is called Weyl rescaling and henceforth assumed to be of fundamental importance. This new principle would forbid me adding a cosmological term $\int \mathrm{d}\sigma\,\mathrm{d}\tau\,\surd(-\det\tilde{h})$ to (8) but would allow an Einstein term

$$A(\text{Einstein}) = \text{const.}\iint \mathrm{d}\sigma\,\mathrm{d}\tau\,\surd(-\det\tilde{h})\,R, \tag{12}$$

where R is the scalar curvature. Because the world-sheet is two dimensional, the integrand in (12) is a total derivative. As a consequence the equations of motion already considered would be unaffected by the addition of (12) to (8).

However, it could have important global effects. So far, I have mentioned strings with ends but these imply that I must also have strings with no ends, i.e. closed strings. Their world-sheets have no boundary and are easier to consider. A freely propagating closed string would produce a world-sheet like a cylinder, see figure 3. But I could consider a surface with more complicated topology, as in figure 4, with g holes. Physically, this describes a string that splits

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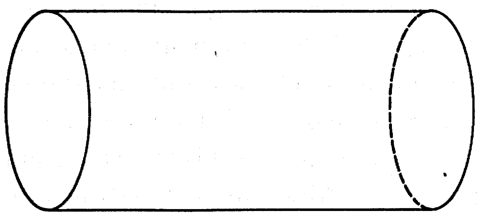

FIGURE 3

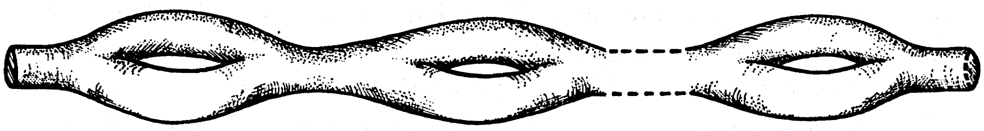

FIGURE 4

and recombines g times by means of $2g$ vertices. In this way the string manifests interaction. The contributions to the Feynman integral (3) from such surfaces all have a common factor

$$e^{iA(\text{Einstein})} = e^{i\,\text{const.}(\text{Euler number})} = e^{i\,\text{const.}(2-2g)},$$

by the Gauss–Bonnet theorem, as A(Einstein) is topological. Thus all world-sheets with the same topology contribute a common factor raised to the power of the number of interactions via the constant in (12). This is the interaction or coupling constant.

We see that string theory possesses a simple geometrical mechanism for describing interaction by means of a complication of the topology of the world-sheet through the addition of holes. This mechanism would seem unlikely to produce divergences because there is no apparent singular point of the type that we saw was responsible for the divergences in the point-particle situation. Yet the successes of the gauge theories could also be encompassed if we could understand the choice of gauge group.

The discussion so far is incomplete in that the absence of anomalies in quantum mechanics must be properly checked. This can be done in at least two different ways leading to the same conclusions: (1) the open string spectrum contains a massless spin one particle, i.e. a gauge particle, (2) the closed string spectrum contains a massless spin two particle, i.e. the graviton, the carrier of gravitational forces, and (3) the theory avoids anomalies in Lorentz covariance only if space and time has 26 dimensions!

The first two results are highly satisfactory in that they provide more evidence that string theory does indeed include the gauge and gravitational forces. Some greeted the third result with consternation while others regarded it as the first indication of exceptional structure. It was Lovelace (1971) who was the first to realize that there had to be $26-2 = 24$ transverse dimensions to facilitate the action of an element of the modular group that related closed string states to open string states. The exceptional structure of which this was suggestive is the special even self-dual lattice in 24 dimensions, the Leech lattice. Now we know that there is a holomorphic conformal field theory related to this whose symmetry group is the famous monster group (Frenkel *et al.* 1988).

The string theory described so far has the serious drawback that the particle states are all bosonic and never fermionic. Hence the theory lacks leptons and quarks, an essential ingredient for a would-be unified theory. A variant of string theory including fermions was initiated by

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Ramond (1971) and by Neveu & Schwarz (1971) and developed in the years following. It possessed supersymmetry on the world-sheet (and stimulated much activity in that subject). For consistency it required 10 dimensions of space and time rather than 26, but, like the purely bosonic theory, possessed a particle moving faster than light and thus unacceptable. The fermionic string theory had the advantage that this tachyon could be eliminated in a way that not only respected the existing symmetries, but enlarged them to manifest supersymmetry in space and time (Gliozzi *et al.* 1977). As a result this theory was later called the 'superstring' (Green & Schwarz 1981). The construction of this supersymmetry algebra owed much to the fact that the transverse space has the same dimension, $10-2 = 8$, as the algebra of octonions. Indeed, the exceptional Jordan algebra of hermitian three by three matrices with octonion entries plays a role and I sense that the trail of exceptional structures is growing even warmer.

It is possible to add gauge symmetry to such a theory, but then one has to check the absence of an anomaly for it in 10 dimensions. There the representations carried by left-handed and right-handed fermionic matter are separately real and unrelated. In fact, by supersymmetry, one of these representations is trivial and the other adjoint to match the gauge particles. Green & Schwarz found that, given this, the anomaly cancelled only if the gauge group was $E_8 \times E_8$ or SO(32). The easiest way to understand this dramatic result is to note that both these possess rank 16 and weight lattices (or sublattices) that are even and self-dual, properties good for constructing theta functions with simple behaviour under the action of the modular group (an important feature of a consistent string theory). The first possible choice of group is close to explaining the exceptional group structures observed in the sequence (2).

A superstring theory with this gauge symmetry was found by the 'heterotic' construction, so called because it awkwardly matched a 26-dimensional left-handed bosonic string with a 10-dimensional right-handed superstring (Gross *et al.* 1985). The 16 excess dimensions of the bosonic string moved on a maximal torus of either of the stated groups, thereby yielding that group as a gauge symmetry according to the vertex operator construction of Frenkel & Kac (1980) and of Segal (1981).

This leaves the problem of reducing from 10 to four dimensions, or of moving in some more direct manner with an explanation of the symmetry-breaking effects mentioned earlier. There is much activity, but as yet no satisfactory and compelling solution. Peter Goddard, Graham Ross and John Schwarz discuss the latest progress in their papers in this Symposium.

The question of divergences is still open, with no universally accepted proof of their absence. S. Mandelstam (unpublished work) has announced a proof for superstring theory but, as far as I know, no other workers have been able to examine the details of this proof yet. This result would be of considerable importance as it would establish, for the first time, a finite theory of gravity.

I can now say something about the quantization of the string motion. This can be done algebraically, forming the hamiltonian and using canonical quantization rules. An important role is played by the energy–momentum tensor (10), which is traceless in the sense that $h^{\alpha\beta}\theta_{\alpha\beta} = 0$, by definition, without recourse to (9). This tensor has only two components, which when appropriately chosen depend only on σ and τ, respectively. In a suitable basis these two components generate two commuting copies of the Virasoro algebra:

$$[L_m, L_n] = (m-n)\, L_{m+n} + \tfrac{1}{12} D\, m(m^2-1)\, \delta_{m+n,0}. \tag{13}$$

The non-zero term on the right-hand side of (13) means that (9) cannot hold quantum

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mechanically (unless Faddeev–Popov ghosts are introduced and $D = 26$). Instead one considers 'physical states' corresponding to physical particles satisfying

$$(L_m - \delta_{m,0})\,|\text{phys}\rangle = 0. \tag{14}$$

Because of time-like oscillations the Hilbert space of quantized oscillations is not positive definite. However, the subspace satisfying (14) is positive definite when a subspace of zero-norm states is modded out. This is the celebrated 'no ghost' theorem of Brower (1972) and of Goddard & Thorn (1972), valid providing D is less than or equal to 26. The structure of this proof has lead to a representation theory of the Virasoro algebra (13) for general values of D not necessarily an integer. This theory is used for classifying more general theories with conformal symmetry and in particular, gives the critical exponents that govern the power law behaviour of correlation functions in two-dimensional materials making second-order phase transitions (Belavin *et al.* 1984; Friedan *et al.* 1984). This exciting application of string theory has developed rapidly with affine Kac–Moody algebras playing a key role (Goddard & Olive 1986).

Finally, I briefly mention the more geometric approach to quantization associated with the name of Polyakov (1981). We have seen that the string action (8) is invariant under both diffeomorphisms and Weyl rescalings of the metric $\tilde{h}_{\alpha\beta}$. Thus in the integration over metrics implied by the Feynman integral (3), I should only integrate over conformal classes not so related. If, following Polyakov, I imagine performing a 'Wick rotation' in which time becomes imaginary so that both the space-time metric $g^{\lambda\mu}$ and the world-sheet metric $\tilde{h}^{\alpha\beta}$ become positive definite, the conformal classes now label Riemann surfaces. The integration in (3) is that over the moduli space of Riemann surfaces of a given topology (at least for closed strings) and is finite dimensional. However, to obtain this I have to divide out by the volume of the invariance group, a manoeuvre that is customarily performed in quantum field theory by the trick of introducing the Faddeev–Popov ghosts as extra fields (which relate to the cohomology of the Virasoro algebra (13)). The new, combined integration measure is anomaly-free only if $D = 26$ or 10, as appropriate. Obviously, the development of these ideas uses the theory of moduli spaces in an interesting way which is particularly intriguing when it is compared with the algebraic approach previously mentioned.

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Discussion

P. T. Landsberg (*University of Southampton, U.K.*). In cosmology and elsewhere one is driven to talk about the 'coupling up' of the extra dimensions. Can Professor Olive comment on this problem.

D. I. Olive, F.R.S. I presume Professor Landsberg is referring to the 'curling up' of surplus dimensions of space whereby the conjugate momenta appear as internal charges. In string theory, as opposed to simpler theories, there is a more sophisticated and comprehensive version that I referred to when I mentioned the vertex operator construction for affine Kac–Moody algebras. I expect that other contributors, namely Goddard, Schwarz & Ross, have more to say about this.

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